Construction d’un modèle réduit pour les problèmes à interfaces (parfois dénommés ‘écoulements à surface libre’)

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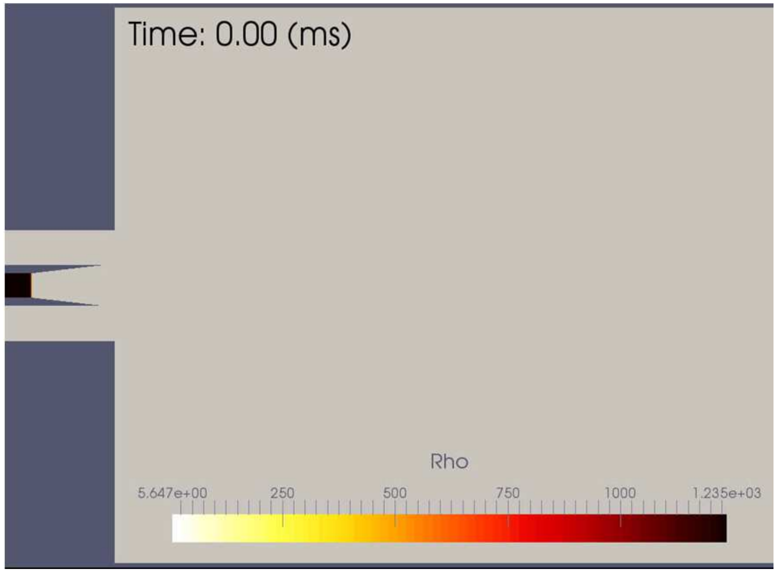
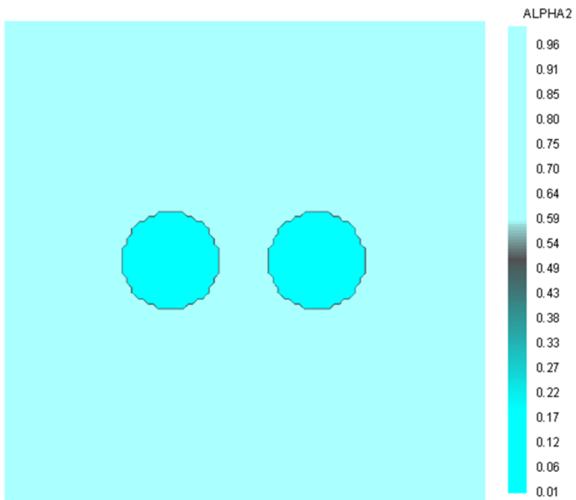
Exemples

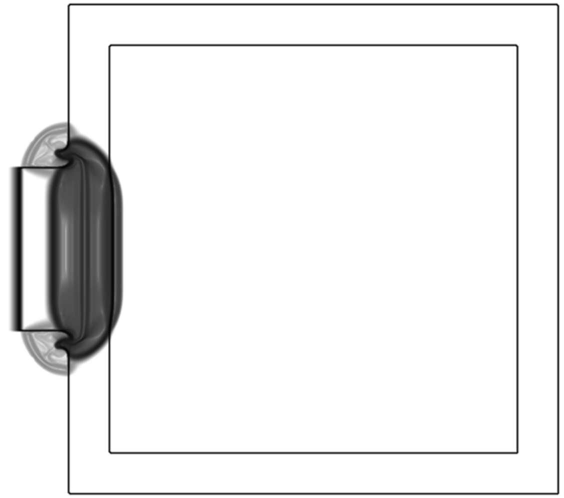
1) Déferlement de vagues en génie côtier.
2) Atomisation d'un jet liquide.



3) Fragmentation et impact de gouttes.



4) Impact hypervéloce



5) Cavitation dans les tuyères et pompes

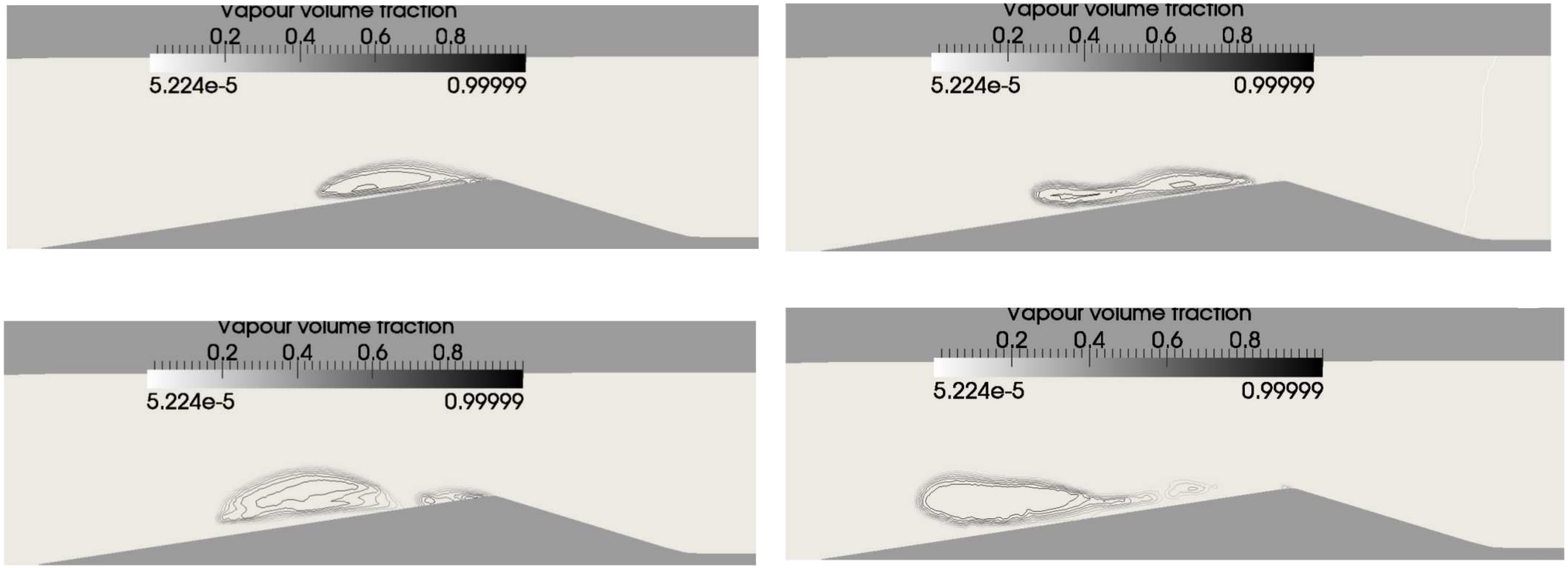
a)

b)

c)

d)

sens de l'écoulement



Résultats de simulation

Modèle réduit

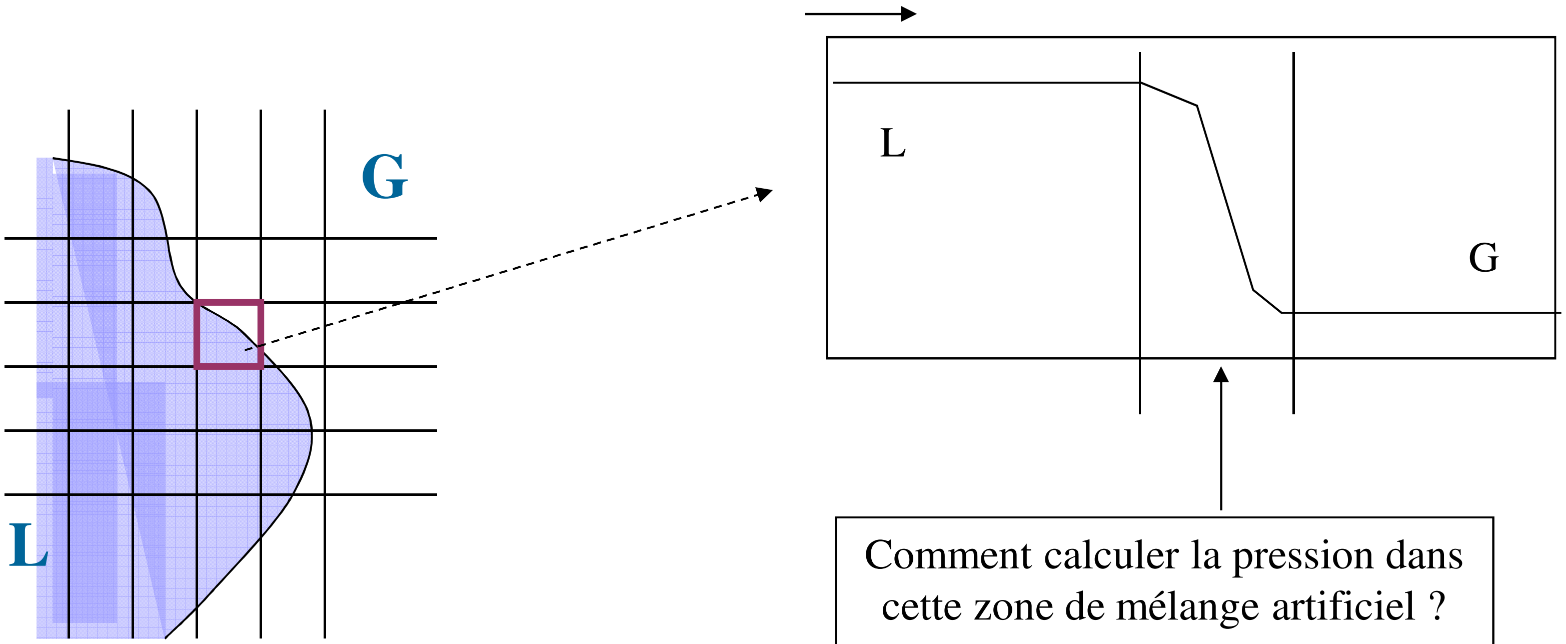
On cherche un modèle en équilibre mécanique:

$$p_1 = p_2 \qquad u_1 = u_2$$

dans le but de résoudre des problèmes à interfaces.

En effet, la condition d'interface est précisément p=cste et u=cste → cf conditions d'interface du 2eme cours dans le cas particulier $\sigma = u$

Mailles mixtes



Comment calculer la pression dans cette zone de mélange artificiel ?

Les équations d'état sont discontinues avec des domaines de validité restreints.

Idée de base

Modèle gaz-gouttes dans lequel on impose à la fois

$\mu \to +\infty$ $\lambda \to +\infty$

$$\frac{\partial \alpha_1}{\partial t} + u_2 \frac{\partial u_1}{\partial x} = \mu (p_1 - p_2)$$

$$\frac{\partial \alpha_1 \rho_1}{\partial t} + \frac{\partial \alpha_1 \rho_1 u_1}{\partial x} = 0$$

$$\frac{\partial \alpha_1 \rho_1 u_1}{\partial t} + \frac{\partial \alpha_1 \rho_1 u_1^2 + \alpha_1 p_1}{\partial x} = p_1 \frac{\partial \alpha_1}{\partial x} + \lambda (u_2 - u_1)$$

$$\frac{\partial \alpha_1 \rho_1 E_1}{\partial t} + \frac{\partial \alpha_1 \rho_1 E_1 u_1 + \alpha_1 p_1 u_1}{\partial x} = p_1 u_2 \frac{\partial \alpha_1}{\partial x} - p_1 \mu (p_1 - p_2) + H\left(T_2 - T_1\right) + u_2 \lambda (u_2 - u_1)$$

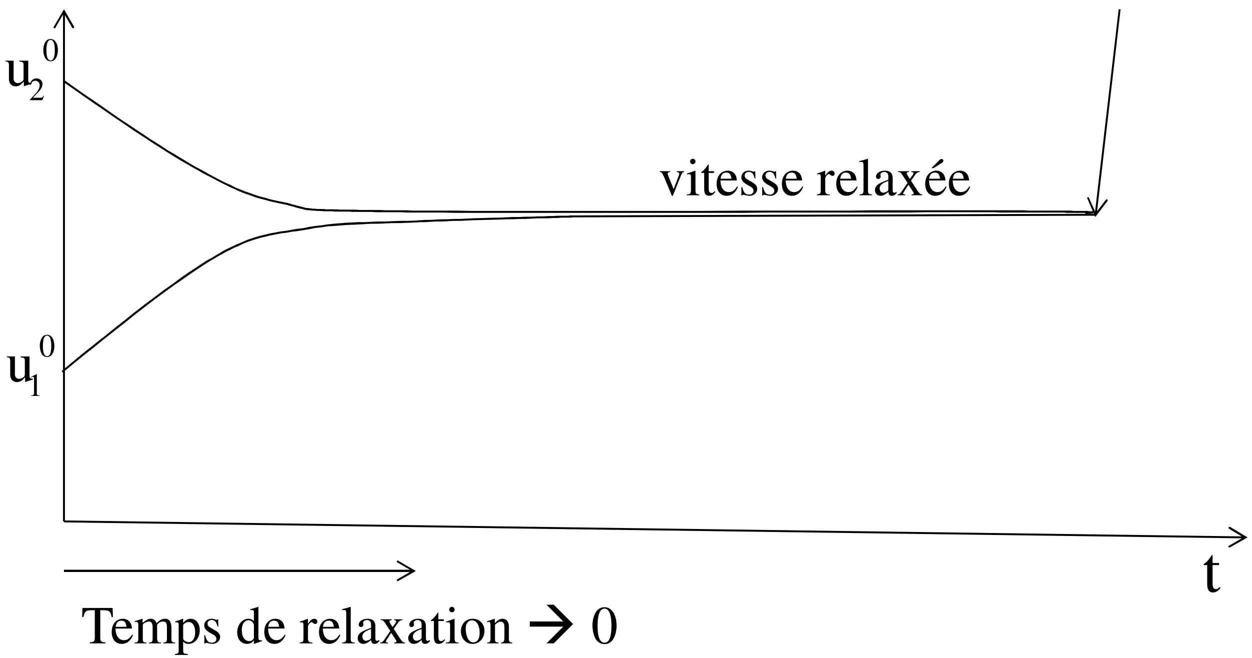
$$\frac{\partial \alpha_2 \rho_2}{\partial t} + \frac{\partial \alpha_2 \rho_2 u_2}{\partial x} = 0$$

$$\frac{\partial \alpha_2 \rho_2 u_2}{\partial t} + \frac{\partial \alpha_2 \rho_2 u_2^2 + \alpha_2 p_2}{\partial x} = p_1 \frac{\partial \alpha_2}{\partial x} + \lambda (u_1 - u_2)$$

$$\frac{\partial \alpha_2 \rho_2 E_2}{\partial t} + \frac{\partial \alpha_2 \rho_2 E_2 u_2 + \alpha_2 p_2 u_2}{\partial x} = -p_1 \frac{\partial \alpha_1}{\partial x} + p_1 \mu (p_1 - p_2) + H(T_1 - T_2) + u_2 \lambda (u_1 - u_2)$$

Les mailles de mélange à l'interface, qui résultent de la diffusion numérique, vont être traitées comme un mélange diphasique avec frottement rapide et relaxation des pressions.

$$u^* = \frac{(\alpha_1\rho_1 u_1 + \alpha_2\rho_2 u_2)^0}{(\alpha_1\rho_1 + \alpha_2\rho_2)^*} = \frac{(\alpha_1\rho_1 u_1 + \alpha_2\rho_2 u_2)^0}{(\alpha_1\rho_1 + \alpha_2\rho_2)^0}$$



- Première option

Résoudre le modèle à 7 équations partout puis dans la zone interfaciale relaxer les pressions et les vitesses: écraser les valeurs calculées par les valeurs d’équilibre.

- Deuxième option

Chercher un système d’équations plus simple.

Analyse asymptotique

1) On écrit le système à 7 équations en variables ‘primitives’:

$$W=(\alpha_1,\rho_1,\rho_2,u_1,u_2,p_1,p_2)^T$$

2) On écrit chaque variable suivant un développement de Taylor d’ordre 1:

$$f = f^{o} + \varepsilon f^{1}$$

$$\lambda,\mu = 1/\varepsilon \rightarrow +\infty$$

3) On simplifie … ce que l’on peut simplifier.

$$\frac{\partial \alpha_1}{\partial t} + u_2 \frac{\partial \alpha_1}{\partial x} = \mu(p_1 - p_2)$$

$$\frac{\partial \rho_1}{\partial t} + u_1 \frac{\partial \rho_1}{\partial x} + \frac{\rho_1 (u_1 - u_2)}{\alpha_1} \frac{\partial \alpha_1}{\partial x} + \rho_1 \frac{\partial u_1}{\partial x} = -\frac{\rho_1}{\alpha_1} \mu(p_1 - p_2)$$

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + \frac{1}{\rho_1} \frac{\partial p_1}{\partial x} = \frac{\lambda(u_2 - u_1)}{\alpha_1 \rho_1}$$

$$\alpha_1 \rho_1 \left(\frac{\partial e_1}{\partial t} + u_1 \frac{\partial e_1}{\partial x} \right) + p_1 (u_1 - u_2) \frac{\partial \alpha_1}{\partial x} + \alpha_1 p_1 \frac{\partial u_1}{\partial x} = \lambda(u_2 - u_1)^2 - p_I \mu(p_1 - p_2) + H(T_2 - T_1)$$

Il faut déduire l’équation sur la pression de cette équation d’énergie:

$$e_1 = e_1(p_1, \rho_1)$$

$$\alpha_1 \rho_1 \left.\frac{\partial e_1}{\partial \rho_1}\right)_{p_1} \left(\frac{\partial \rho_1}{\partial t} + u_1 \frac{\partial \rho_1}{\partial x} \right) + \alpha_1 \rho_1 \left.\frac{\partial e_1}{\partial p_1}\right)_{\rho_1} \left(\frac{\partial p_1}{\partial t} + u_1 \frac{\partial p_1}{\partial x} \right) + p_1 (u_1 - u_2) \frac{\partial \alpha_1}{\partial x} + \alpha_1 p_1 \frac{\partial u_1}{\partial x}$$

$$= \lambda(u_2 - u_1)^2 - p_1 \mu(p_1 - p_2) + H(T_2 - T_1)$$

$$\alpha_1\rho_1 \left.\frac{\partial e_1}{\partial \rho_1}\right)_{p_1}\left(\frac{\partial \rho_1}{\partial t}+u_1\frac{\partial \rho_1}{\partial x}\right)+\alpha_1\rho_1\left.\frac{\partial e_1}{\partial p_1}\right)_{\rho_1}\left(\frac{\partial p_1}{\partial t}+u_1\frac{\partial p_1}{\partial x}\right)+p_1\left(u_1-u_2\right)\frac{\partial \alpha_1}{\partial x}+\alpha_1 p_1\frac{\partial u_1}{\partial x}$$

$$=\lambda(u_2-u_1)^2-p_1\mu(p_1-p_2)+H(T_2-T_1)$$

$$\alpha_1\rho_1 \left.\frac{\partial e_1}{\partial \rho_1}\right)_{p_1}\left(-\frac{\rho_1\left(u_1-u_2\right)}{\alpha_1}\frac{\partial \alpha_1}{\partial x}-\rho_1\frac{\partial u_1}{\partial x}-\frac{\rho_1}{\alpha_1}\mu(p_1-p_2)\right)+\alpha_1\rho_1\left.\frac{\partial e_1}{\partial p_1}\right)_{\rho_1}\left(\frac{\partial p_1}{\partial t}+u_1\frac{\partial p_1}{\partial x}\right)$$

$$+p_1\left(u_1-u_2\right)\frac{\partial \alpha_1}{\partial x}+\alpha_1 p_1\frac{\partial u_1}{\partial x}=\lambda(u_2-u_1)^2-p_1\mu(p_1-p_2)+H(T_2-T_1)$$

$$\alpha_1\rho_1\left(\frac{\partial p_1}{\partial t}+u_1\frac{\partial p_1}{\partial x}\right)+\left(\frac{p_1-\rho_1^2\left.\frac{\partial e_1}{\partial \rho_1}\right)_{p_1}}{\left.\frac{\partial e_1}{\partial p_1}\right)_{\rho_1}}\right)\left(u_1-u_2\right)\frac{\partial \alpha_1}{\partial x}+\alpha_1\left(\frac{p_1-\rho_1^2\left.\frac{\partial e_1}{\partial \rho_1}\right)_{p_1}}{\left.\frac{\partial e_1}{\partial p_1}\right)_{\rho_1}}\right)\frac{\partial u_1}{\partial x}$$

$$=\frac{\lambda(u_2-u_1)^2}{\left.\frac{\partial e_1}{\partial p_1}\right)_{\rho_1}}-\mu(p_1-p_2)\left(\frac{p_1-\rho_1^2\left.\frac{\partial e_1}{\partial \rho_1}\right)_{p_1}}{\left.\frac{\partial e_1}{\partial p_1}\right)_{\rho_1}}\right)+\frac{H(T_2-T_1)}{\left.\frac{\partial e_1}{\partial p_1}\right)_{\rho_1}}$$

$$\alpha_1\rho_1\left(\frac{\partial p_1}{\partial t}+u_1\frac{\partial p_1}{\partial x}\right)+\left(\frac{p_1-\rho_1^2\left.\frac{\partial e_1}{\partial \rho_1}\right)_{p_1}}{\left.\frac{\partial e_1}{\partial p_1}\right)_{\rho_1}}\right)(u_1-u_2)\frac{\partial \alpha_1}{\partial x}+\alpha_1\left(\frac{p_1-\rho_1^2\left.\frac{\partial e_1}{\partial \rho_1}\right)_{p_1}}{\left.\frac{\partial e_1}{\partial p_1}\right)_{\rho_1}}\right)\frac{\partial u_1}{\partial x}$$

$$=\frac{\lambda(u_2-u_1)^2}{\left.\frac{\partial e_1}{\partial p_1}\right)_{\rho_1}}-\mu(p_1-p_2)\left(\frac{p_1-\rho_1^2\left.\frac{\partial e_1}{\partial \rho_1}\right)_{p_1}}{\left.\frac{\partial e_1}{\partial p_1}\right)_{\rho_1}}\right)+\frac{H(T_2-T_1)}{\left.\frac{\partial e_1}{\partial p_1}\right)_{\rho_1}}$$

$$de = Tds - pdv \qquad de = -pdv \qquad \left.\frac{\partial e}{\partial p}\right)_{\rho} dp + \left.\frac{\partial e}{\partial \rho}\right)_{p} d\rho = \frac{p}{\rho^2}d\rho$$

$$c^2 = \left.\frac{\partial p}{\partial \rho}\right)_s$$

$$\left.\frac{\partial e}{\partial p}\right)_{\rho} dp = \left(\frac{p}{\rho^2} - \left.\frac{\partial e}{\partial \rho}\right)_{p}\right)d\rho$$

$$\frac{dp}{d\rho} = \frac{\frac{p}{\rho^2} - \left.\frac{\partial e}{\partial \rho}\right)_{p}}{\left.\frac{\partial e}{\partial p}\right)_{\rho}} = \left.\frac{\partial p}{\partial \rho}\right)_s = c^2$$

$$\alpha_1\rho_1\left(\frac{\partial p_1}{\partial t}+u_1\frac{\partial p_1}{\partial x}\right)+\rho_1^2c_1^2\left(u_1-u_2\right)\frac{\partial\alpha_1}{\partial x}+\alpha_1\rho_1^2c_1^2\frac{\partial u_1}{\partial x}=\frac{\lambda(u_2-u_1)^2}{\left.\frac{\partial e_1}{\partial p_1}\right)_{\rho_1}}-\mu(p_1-p_2)\rho_1^2c_1^2+\frac{H(T_2-T_1)}{\left.\frac{\partial e_1}{\partial p_1}\right)_{\rho_1}}$$

$$\frac{\partial p_1}{\partial t}+u_1\frac{\partial p_1}{\partial x}+\frac{\rho_1c_1^2}{\alpha_1}\left(u_1-u_2\right)\frac{\partial\alpha_1}{\partial x}+\rho_1c_1^2\frac{\partial u_1}{\partial x}=\frac{\lambda(u_2-u_1)^2}{\alpha_1\rho_1\left.\frac{\partial e_1}{\partial p_1}\right)_{\rho_1}}-\frac{\mu(p_1-p_2)\rho_1c_1^2}{\alpha_1}+\frac{H(T_2-T_1)}{\alpha_1\rho_1\left.\frac{\partial e_1}{\partial p_1}\right)_{\rho_1}}$$

OK

Equations en variables primitives (ou physiques)

$$\frac{\partial\alpha_1}{\partial t}+u_2\frac{\partial\alpha_1}{\partial x}=\mu(p_1-p_2)$$

$$\frac{\partial\rho_1}{\partial t}+u_1\frac{\partial\rho_1}{\partial x}+\frac{\rho_1\left(u_1-u_2\right)}{\alpha_1}\frac{\partial\alpha_1}{\partial x}+\rho_1\frac{\partial u_1}{\partial x}=-\frac{\rho_1}{\alpha_1}\mu(p_1-p_2)$$

$$\frac{\partial u_1}{\partial t}+u_1\frac{\partial u_1}{\partial x}+\frac{1}{\rho_1}\frac{\partial p_1}{\partial x}=\frac{\lambda(u_2-u_1)}{\alpha_1\rho_1}$$

$$\frac{\partial p_1}{\partial t}+u_1\frac{\partial p_1}{\partial x}+\frac{\rho_1c_1^2}{\alpha_1}\left(u_1-u_2\right)\frac{\partial\alpha_1}{\partial x}+\rho_1c_1^2\frac{\partial u_1}{\partial x}=\frac{\lambda(u_2-u_1)^2}{\alpha_1\rho_1\left.\frac{\partial e_1}{\partial p_1}\right)_{\rho_1}}-\frac{\mu(p_1-p_2)\rho_1c_1^2}{\alpha_1}+\frac{H(T_2-T_1)}{\alpha_1\rho_1\left.\frac{\partial e_1}{\partial p_1}\right)_{\rho_1}}$$

$$\frac{\partial\rho_2}{\partial t}+u_2\frac{\partial\rho_2}{\partial x}+\frac{\rho_2\left(u_2-u_1\right)}{\alpha_2}\frac{\partial\alpha_2}{\partial x}+\rho_2\frac{\partial u_2}{\partial x}=-\frac{\rho_2}{\alpha_2}\mu(p_2-p_1)$$

$$\frac{\partial u_2}{\partial t}+u_2\frac{\partial u_2}{\partial x}+\frac{1}{\rho_2}\frac{\partial p_2}{\partial x}=\frac{\lambda(u_1-u_2)}{\alpha_2\rho_2}$$

$$\frac{\partial p_2}{\partial t}+u_2\frac{\partial p_2}{\partial x}+\frac{\rho_2c_2^2}{\alpha_2}\left(u_2-u_1\right)\frac{\partial\alpha_2}{\partial x}+\rho_2c_2^2\frac{\partial u_2}{\partial x}=\frac{\lambda(u_2-u_1)^2}{\alpha_2\rho_2\left.\frac{\partial e_2}{\partial p_2}\right)_{\rho_2}}-\frac{\mu(p_2-p_1)\rho_2c_2^2}{\alpha_2}+\frac{H(T_1-T_2)}{\alpha_2\rho_2\left.\frac{\partial e_2}{\partial p_2}\right)_{\rho_2}}$$

Analyse asymptotique de la première équation

$$\frac{\partial \alpha_1}{\partial t} + u_2 \frac{\partial \alpha_1}{\partial x} = \mu(p_1 - p_2)$$

$$f = f^o + \varepsilon f^1 \qquad \lambda, \mu = 1/\varepsilon \to +\infty$$

donc $\alpha_1 = \alpha_1^0 + \varepsilon \alpha_1^1 \,;\, u_2 = u_2^0 + \varepsilon u_2^1 \,;\, p_k = p_k^0 + \varepsilon p_k^1$

$$\frac{\partial \alpha_1^0 + \varepsilon \alpha_1^1}{\partial t} + \left(u_2^0 + \varepsilon u_2^1\right) \frac{\partial \alpha_1^0 + \varepsilon \alpha_1^1}{\partial x} = \frac{1}{\varepsilon}(p_1^0 + \varepsilon p_1^1 - p_2^0 - \varepsilon p_2^1)$$

on ordonne ceci comme un polynome en ε

$$\frac{\partial \alpha_1^0}{\partial t} + u_2^0 \frac{\partial \alpha_1^0}{\partial x} + \varepsilon \left(\frac{\partial \alpha_1^1}{\partial t} + u_2^0 \frac{\partial \alpha_1^1}{\partial x} + u_2^1 \frac{\partial \alpha_1^0}{\partial x} \right) + \varepsilon^2 u_2^1 \frac{\partial \alpha_1^1}{\partial x} = \frac{p_1^0 - p_2^0}{\varepsilon} + (p_1^1 - p_2^1)$$

comme ε peut prendre des valeurs arbitrairement petites il faut que les 4 équations suivantes soient satisfaites:

$$\frac{p_1^0 - p_2^0}{\varepsilon} = 0 \qquad \text{ordre } 1/\varepsilon$$

$$\frac{\partial \alpha_1^0}{\partial t} + u_2^0 \frac{\partial \alpha_1^0}{\partial x} = (p_1^1 - p_2^1) \qquad \text{ordre 0 ou ordre dominant}$$

$$\frac{\partial \alpha_1^1}{\partial t} + u_2^0 \frac{\partial \alpha_1^1}{\partial x} + u_2^1 \frac{\partial \alpha_1^0}{\partial x} = 0 \qquad \text{ordre } \varepsilon$$

$$u_2^1 \frac{\partial \alpha_1^1}{\partial x} = 0 \qquad \text{ordre } \varepsilon^2$$

Deux équations particulièrement importantes

$$\frac{p_1^0 - p_2^0}{\varepsilon} = 0$$

ordre 1/ε - Comme ε est arbitrairement petit, il n'y a pas d'autre choix que:

$$p_1^0 - p_2^0 = 0 \qquad p_1^0 = p_2^0 = p^0$$

les pressions sont égales à l'ordre dominant

$$\frac{\partial \alpha_1^0}{\partial t} + u_2^0 \frac{\partial \alpha_1^0}{\partial x} = (p_1^1 - p_2^1)$$

ordre 0 ou ordre dominant

Un terme de fluctuation des pressions reste présent. Il va falloir le déterminer.

Examinons les équations sur les vitesses

$$\frac{\partial u_1^0+\varepsilon u_1^1}{\partial t}+(u_1^0+\varepsilon u_1^1)\frac{\partial u_1^0+\varepsilon u_1^1}{\partial x}+\frac{1}{\rho_1^0+\varepsilon\rho_1^1}\frac{\partial p_1^0+\varepsilon p_1^1}{\partial x}=\frac{\lambda(u_2^0+\varepsilon u_2^1-u_1^0-\varepsilon u_1^1)}{(\alpha_1^0+\varepsilon\alpha_1^1)(\rho_1^0+\varepsilon\rho_1^1)}$$

Cherchons seulement les équations aux ordre 0 et ordre $1/\varepsilon$.

$$\frac{\partial u_1^0+\varepsilon u_1^1}{\partial t}+(u_1^0+\varepsilon u_1^1)\frac{\partial u_1^0+\varepsilon u_1^1}{\partial x}+\frac{1}{\rho_1^0+\varepsilon\rho_1^1}\frac{\partial p_1^0+\varepsilon p_1^1}{\partial x}=\frac{\lambda(u_2^0+\varepsilon u_2^1-u_1^0-\varepsilon u_1^1)}{(\alpha_1^0+\varepsilon\alpha_1^1)(\rho_1^0+\varepsilon\rho_1^1)}$$

$$\frac{u_2^0-u_1^0}{\varepsilon\alpha_1^0\rho_1^0}=0 \longrightarrow u_2^0=u_1^0=u^0$$

$$\frac{\partial u_1^0}{\partial t}+u_1^0\frac{\partial u_1^0}{\partial x}+\frac{1}{\rho_1^0}\frac{\partial p_1^0}{\partial x}=\frac{(u_2^1-u_1^1)}{\alpha_1^0\rho_1^0}$$

$$\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+\frac{1}{\rho_1^0}\frac{\partial p^0}{\partial x}=\frac{(u_2^1-u_1^1)}{\alpha_1^0\rho_1^0}$$

qdm

$$\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+\frac{1}{\rho_1^0}\frac{\partial p^0}{\partial x}=\frac{(u_2^1-u_1^1)}{\alpha_1^0\rho_1^0}$$ phase 1

$$\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+\frac{1}{\rho_2^0}\frac{\partial p^0}{\partial x}=\frac{(u_1^1-u_2^1)}{\alpha_2^0\rho_2^0}$$ phase 2

Somme:

$$\alpha_1^0\rho_1^0\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+\frac{1}{\rho_1^0}\frac{\partial p^0}{\partial x}\right)+\alpha_2^0\rho_2^0\left(\frac{\partial u^0}{\partial t}+u^0\frac{\partial u^0}{\partial x}+\frac{1}{\rho_2^0}\frac{\partial p^0}{\partial x}\right)=0$$

$$\left(\alpha_1^0\rho_1^0+\alpha_2^0\rho_2^0\right)\frac{\partial u^0}{\partial t}+\left(\alpha_1^0\rho_1^0+\alpha_2^0\rho_2^0\right)u^0\frac{\partial u^0}{\partial x}+\left(\alpha_1^0+\alpha_2^0\right)\frac{\partial p^0}{\partial x}=0$$

$$\rho^0=\alpha_1^0\rho_1^0+\alpha_2^0\rho_2^0$$

$$\rho^0\frac{\partial u^0}{\partial t}+\rho^0u^0\frac{\partial u^0}{\partial x}+\frac{\partial p^0}{\partial x}=0$$

$$\frac{\partial \rho^0u^0}{\partial t}-u^0\frac{\partial \rho^0}{\partial t}+\frac{\partial \rho^0u^{02}}{\partial x}-u^0\frac{\partial \rho^0u^0}{\partial x}+\frac{\partial p^0}{\partial x}=0$$

$$\boxed{\frac{\partial \rho^0u^0}{\partial t}+\frac{\partial \rho^0u^{02}+p^0}{\partial x}=0}$$

Puisque les phases évoluent avec la même vitesse et la même pression, ceci est assez normal.

Différence de fluctuation des pressions

$$\frac{\partial \alpha_1^0}{\partial t} + u_2^0 \frac{\partial \alpha_1^0}{\partial x} = (p_1^1 - p_2^1)$$

On dispose des équations sur les pressions:

$$\frac{\partial p_1}{\partial t} + u_1 \frac{\partial p_1}{\partial x} + \frac{\rho_1 c_1^2}{\alpha_1}\left(u_1 - u_2\right)\frac{\partial \alpha_1}{\partial x} + \rho_1 c_1^2 \frac{\partial u_1}{\partial x} = \frac{\lambda (u_2 - u_1)^2}{\alpha_1 \rho_1 \left.\frac{\partial e_1}{\partial p_1}\right)_{\rho_1}} - \frac{\mu (p_1 - p_2)\rho_1 c_1^2}{\alpha_1} + \frac{H(T_2 - T_1)}{\alpha_1 \rho_1 \left.\frac{\partial e_1}{\partial p_1}\right)_{\rho_1}}$$

$$\frac{\partial p_2}{\partial t} + u_2 \frac{\partial p_2}{\partial x} + \frac{\rho_2 c_2^2}{\alpha_2}\left(u_2 - u_1\right)\frac{\partial \alpha_2}{\partial x} + \rho_2 c_2^2 \frac{\partial u_2}{\partial x} = \frac{\lambda (u_2 - u_1)^2}{\alpha_2 \rho_2 \left.\frac{\partial e_2}{\partial p_2}\right)_{\rho_2}} - \frac{\mu (p_2 - p_1)\rho_2 c_2^2}{\alpha_2} + \frac{H(T_1 - T_2)}{\alpha_2 \rho_2 \left.\frac{\partial e_2}{\partial p_2}\right)_{\rho_2}}$$

A l'ordre dominant elles deviennent:

$$\frac{\partial p^0}{\partial t} + u^0 \frac{\partial p^0}{\partial x} + \rho_1^0 c_1^{02} \frac{\partial u^0}{\partial x} = -\frac{(p_1^1 - p_2^1)\rho_1^0 c_1^{02}}{\alpha_1^0} + \frac{H(T_2^0 - T_1^0)}{\alpha_1^0 \rho_1^0 \left.\frac{\partial e_1}{\partial p_1}\right)_{\rho_1}^0}$$

Prenons la différence de ces deux équations

$$\frac{\partial p^0}{\partial t} + u^0 \frac{\partial p^0}{\partial x} + \rho_2^0 c_2^{02} \frac{\partial u^0}{\partial x} = -\frac{(p_2^1 - p_1^1)\rho_2^0 c_2^{02}}{\alpha_2^0} + \frac{H(T_1^0 - T_2^0)}{\alpha_2^0 \rho_2^0 \left.\frac{\partial e_2}{\partial p_2}\right)_{\rho_2}^0}$$

$$\frac{\partial p^0}{\partial t}+u^0\frac{\partial p^0}{\partial x}+\rho_1^0 c_1^{02}\frac{\partial u^0}{\partial x}=-\frac{(p_1^1-p_2^1)\rho_1^0 c_1^{02}}{\alpha_1^0}+\frac{H(T_2^0-T_1^0)}{\alpha_1^0\rho_1^0\left.\frac{\partial e_1}{\partial p_1}\right)_{\rho_1}^0}$$

$$\frac{\partial p^0}{\partial t}+u^0\frac{\partial p^0}{\partial x}+\rho_2^0 c_2^{02}\frac{\partial u^0}{\partial x}=-\frac{(p_2^1-p_1^1)\rho_2^0 c_2^{02}}{\alpha_2^0}+\frac{H(T_1^0-T_2^0)}{\alpha_2^0\rho_2^0\left.\frac{\partial e_2}{\partial p_2}\right)_{\rho_2}^0}$$

$$(\rho_1^0 c_1^{02}-\rho_2^0 c_2^{02})\frac{\partial u^0}{\partial x}=-\frac{(p_1^1-p_2^1)\rho_1^0 c_1^{02}}{\alpha_1^0}+\frac{(p_2^1-p_1^1)\rho_2^0 c_2^{02}}{\alpha_2^0}+\frac{H(T_2^0-T_1^0)}{\alpha_1^0\rho_1^0\left.\frac{\partial e_1}{\partial p_1}\right)_{\rho_1}^0}-\frac{H(T_1^0-T_2^0)}{\alpha_2^0\rho_2^0\left.\frac{\partial e_2}{\partial p_2}\right)_{\rho_2}^0}$$

$$(\rho_1^0 c_1^{02}-\rho_2^0 c_2^{02})\frac{\partial u^0}{\partial x}=-(p_1^1-p_2^1)\left(\frac{\rho_1^0 c_1^{02}}{\alpha_1^0}+\frac{\rho_2^0 c_2^{02}}{\alpha_2^0}\right)+H(T_2^0-T_1^0)\left(\frac{1}{\alpha_1^0\rho_1^0\left.\frac{\partial e_1}{\partial p_1}\right)_{\rho_1}^0}+\frac{1}{\alpha_2^0\rho_2^0\left.\frac{\partial e_2}{\partial p_2}\right)_{\rho_2}^0}\right)$$

$$(\rho_1^0 c_1^{02} - \rho_2^0 c_2^{02}) \frac{\partial u^0}{\partial x} = -(p_1^1 - p_2^1)\left(\frac{\rho_1^0 c_1^{02}}{\alpha_1^0} + \frac{\rho_2^0 c_2^{02}}{\alpha_2^0}\right) + H(T_2^0 - T_1^0)\left(\frac{1}{\alpha_1^0 \rho_1^0 \left.\frac{\partial e_1}{\partial p_1}\right)_{\rho_1}^{0}} + \frac{1}{\alpha_2^0 \rho_2^0 \left.\frac{\partial e_2}{\partial p_2}\right)_{\rho_2}^{0}}\right)$$

$$p_1^1 - p_2^1 = \frac{\rho_2^0 c_2^{02} - \rho_1^0 c_1^{02}}{\frac{\rho_1^0 c_1^{02}}{\alpha_1^0} + \frac{\rho_2^0 c_2^{02}}{\alpha_2^0}} \frac{\partial u^0}{\partial x} + \frac{H(T_2^0 - T_1^0)}{\frac{\rho_1^0 c_1^{02}}{\alpha_1^0} + \frac{\rho_2^0 c_2^{02}}{\alpha_2^0}}\left(\frac{1}{\alpha_1^0 \rho_1^0 \left.\frac{\partial e_1}{\partial p_1}\right)_{\rho_1}^{0}} + \frac{1}{\alpha_2^0 \rho_2^0 \left.\frac{\partial e_2}{\partial p_2}\right)_{\rho_2}^{0}}\right)$$

$$\frac{\partial \alpha_1^0}{\partial t} + u_2^0 \frac{\partial \alpha_1^0}{\partial x} = (p_1^1 - p_2^1)$$

$$\frac{\partial \alpha_1^0}{\partial t} + u_2^0 \frac{\partial \alpha_1^0}{\partial x} = \frac{\rho_2^0 c_2^{02} - \rho_1^0 c_1^{02}}{\frac{\rho_1^0 c_1^{02}}{\alpha_1^0} + \frac{\rho_2^0 c_2^{02}}{\alpha_2^0}} \frac{\partial u^0}{\partial x} + \frac{H(T_2^0 - T_1^0)}{\frac{\rho_1^0 c_1^{02}}{\alpha_1^0} + \frac{\rho_2^0 c_2^{02}}{\alpha_2^0}}\left(\frac{1}{\alpha_1^0 \rho_1^0 \left.\frac{\partial e_1}{\partial p_1}\right)_{\rho_1}^{0}} + \frac{1}{\alpha_2^0 \rho_2^0 \left.\frac{\partial e_2}{\partial p_2}\right)_{\rho_2}^{0}}\right)$$

En résumé

En partant de: $$\frac{\partial \alpha_1}{\partial t} + u_2 \frac{\partial \alpha_1}{\partial x} = \mu(p_1 - p_2)$$

Nous avons obtenu:

$$\frac{\partial \alpha_1}{\partial t} + u \frac{\partial \alpha_1}{\partial x} = \frac{\rho_2 c_2^2 - \rho_1 c_1^2}{\frac{\rho_1 c_1^2}{\alpha_1} + \frac{\rho_2 c_2^2}{\alpha_2}} \frac{\partial u}{\partial x} + \frac{H(T_2 - T_1)}{\frac{\rho_1 c_1^2}{\alpha_1} + \frac{\rho_2 c_2^2}{\alpha_2}} \left(\frac{1}{\alpha_1 \rho_1 \left.\frac{\partial e_1}{\partial p_1}\right)_{\rho_1}} + \frac{1}{\alpha_2 \rho_2 \left.\frac{\partial e_2}{\partial p_2}\right)_{\rho_2}} \right)$$

Un terme source peut donc devenir un terme differentiel.

Le coefficient de Gruneisen est défini par: $$\frac{1}{\Gamma} = \rho \left.\frac{\partial e}{\partial p}\right)_{\rho}$$

$$\frac{\partial \alpha_1}{\partial t} + u \frac{\partial \alpha_1}{\partial x} = \frac{\rho_2 c_2^2 - \rho_1 c_1^2}{\frac{\rho_1 c_1^2}{\alpha_1} + \frac{\rho_2 c_2^2}{\alpha_2}} \frac{\partial u}{\partial x} + H(T_2 - T_1) \frac{\frac{\Gamma_1}{\alpha_1} + \frac{\Gamma_2}{\alpha_2}}{\frac{\rho_1 c_1^2}{\alpha_1} + \frac{\rho_2 c_2^2}{\alpha_2}}$$

Modèle limite en équilibre de pression et de vitesses (Kapila et al., 2001)

$$\frac{\partial \alpha_1}{\partial t} + u\frac{\partial \alpha_1}{\partial x} = \frac{\rho_2 c_2^2 - \rho_1 c_1^2}{\frac{\rho_1 c_1^2}{\alpha_1} + \frac{\rho_2 c_2^2}{\alpha_2}} \frac{\partial u}{\partial x} + H(T_2 - T_1) \frac{\frac{\Gamma_1}{\alpha_1} + \frac{\Gamma_2}{\alpha_2}}{\frac{\rho_1 c_1^2}{\alpha_1} + \frac{\rho_2 c_2^2}{\alpha_2}}$$

$$\frac{\partial \alpha_2 \rho_2}{\partial t} + \frac{\partial \alpha_2 \rho_2 u}{\partial x} = 0$$

$$\frac{\partial \alpha_1 \rho_1}{\partial t} + \frac{\partial \alpha_1 \rho_1 u}{\partial x} = 0$$

$$\frac{\partial \rho u}{\partial t} + \frac{\partial \rho u^2 + p}{\partial x} = 0$$

$$\frac{\partial \rho E}{\partial t} + \frac{\partial (\rho E + p) u}{\partial x} = 0$$

$$\rho = \alpha_1 \rho_1 + \alpha_2 \rho_2$$

$$\rho E = \alpha_1 \rho_1 E_1 + \alpha_2 \rho_2 E_2$$

$$p = p(\alpha_1, \rho, e)$$

Equation d'état du mélange

$$\rho E = \alpha_1 \rho_1 E_1 + \alpha_2 \rho_2 E_2$$

$$\rho e = \alpha_1 \rho_1 e_1 + \alpha_2 \rho_2 e_2$$

$$p_k = (\gamma_k - 1)\rho_k e_k - \gamma_k p_{\infty k}$$

$$\rho_k e_k = \frac{p_k + \gamma_k p_{\infty k}}{\gamma_k - 1}$$

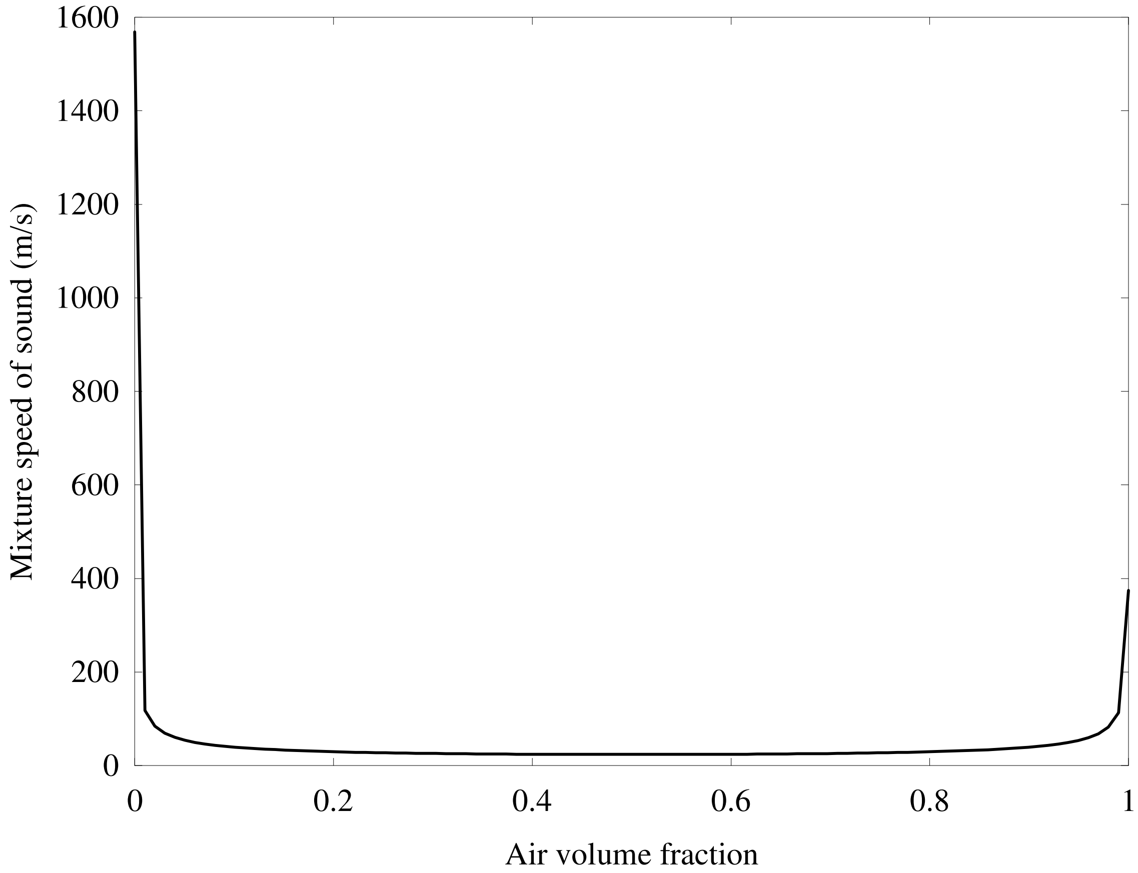
$$\rho e = \alpha_1 \frac{p_1 + \gamma_1 p_{\infty 1}}{\gamma_1 - 1} + \alpha_2 \frac{p_2 + \gamma_2 p_{\infty 2}}{\gamma_2 - 1}$$

$$p_1 = p_2$$

$$\rho e = \left(\frac{\alpha_1}{\gamma_1 - 1} + \frac{\alpha_2}{\gamma_2 - 1} \right) p + \frac{\alpha_1 \gamma_1 p_{\infty 1}}{\gamma_1 - 1} + \frac{\alpha_2 \gamma_2 p_{\infty 2}}{\gamma_2 - 1}$$

$$p(\rho, e, \alpha_1) = \frac{\rho e - \left(\frac{\alpha_1 \gamma_1 p_{\infty 1}}{\gamma_1 - 1} + \frac{\alpha_2 \gamma_2 p_{\infty 2}}{\gamma_2 - 1} \right)}{\frac{\alpha_1}{\gamma_1 - 1} + \frac{\alpha_2}{\gamma_2 - 1}}$$

Propriétés acoustiques et hyperbolicité



Exercice: Montrer que les vitesses d'ondes sont:

$u+c_m$

$u-c_m$

u

→ Les vitesses étant réelles, le modèle est hyperbolique.

La vitesse du son du mélange évolue de façon non monotone avec la fraction volumique, propriété connue depuis Wood (1930) et ici démontrée et validée expérimentalement:

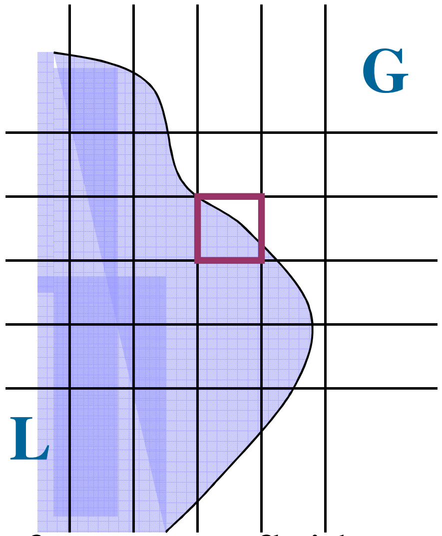
$$\frac{1}{\rho c_m^2} = \frac{\alpha_1}{\rho_1 c_1^2} + \frac{\alpha_2}{\rho_2 c_2^2}$$

Cas limites et mailles de mélange

Lorsque α_1 tend vers 1 on retrouve les équations d'Euler de la phase 1, y compris au niveau de l'équation d'état du mélange qui tend vers celle de la phase 1.

Lorsque α_1 tend vers 0 on retrouve les équations d'Euler de la phase 2, y compris au niveau de l'équation d'état du mélange qui tend vers celle de la phase 2.

Dans la zone de mélange, lorsque les deux phases sont présentes, l'égalité des pressions et des vitesses est assurée:



→ Le modèle capture parfaitement les interfaces entre fluides.

Aspects numériques

- On peut étendre la méthode de Godunov à la résolution du modèle hors d'équilibre (deux vitesses et deux températures).
- On peut en faire de même pour le modèle de Kapila.

Il y a quelques détails non triviaux à traiter.

Objet d'un autre cours….