Relations de fermeture pour le modèle d’écoulement diphasique

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Equations et inconnues

$$\frac{\partial \alpha_k \rho_k}{\partial t} + \frac{\partial \alpha_k \rho_k u_k}{\partial x} = 0$$

$$\frac{\partial \alpha_k \rho_k u_k}{\partial t} + \frac{\partial \alpha_k (\rho_k u_k^2 + p_k)}{\partial x} = p_I \frac{\partial \alpha_k}{\partial x}$$

$$\frac{\partial \alpha_k \rho_k E_k}{\partial t} + \frac{\partial \alpha_k u_k (\rho_k E_k + p_k)}{\partial x} = p_I u_I \frac{\partial \alpha_k}{\partial x}$$

α_1;

Nécessité d'une équation d'évolution supplémentaire. (une seule équation suffit car $\alpha_1 + \alpha_2 = 1$)

ρ_k; u_k; p_k; E_k;

4 inconnues, mais 3 relations + équation d'état.

p_I; u_I.

2 variables à déterminer en fonction des autres variables du dessus.
Le guide que nous allons suivre est la seconde loi de la thermodynamique.

Moyenne de la fonction de phase

$$\frac{\partial X_k}{\partial t} + \vec{\sigma}.\mathrm{grad}(X_k) = 0$$

$$\frac{1}{V}\int_V \left(\frac{\partial X_k}{\partial t} + \vec{\sigma}\mathrm{grad}(X_k) \right) dV = 0$$

Exercice: En utilisant la même démarche que celle qui a été utilisée pour l'équation du mouvement, montrer qu'on obtient:

$$\frac{\partial \alpha_k}{\partial t} + u_I \frac{\partial \alpha_k}{\partial x} = 0$$

Il suffit donc de résoudre:

$$\frac{\partial \alpha_1}{\partial t} + u_I \frac{\partial \alpha_1}{\partial x} = 0$$

On déduit la fraction volumique de la seconde phase par: $\alpha_1 + \alpha_2 = 1$

A ce stade…

$$\frac{\partial \alpha_1}{\partial t} + u_I \frac{\partial \alpha_1}{\partial x} = 0$$

$$\frac{\partial \alpha_1 \rho_1}{\partial t} + \frac{\partial \alpha_1 \rho_1 u_1}{\partial x} = 0$$

$$\frac{\partial \alpha_1 \rho_1 u_1}{\partial t} + \frac{\partial \alpha_1 (\rho_1 u_1^2 + p_1)}{\partial x} = p_I \frac{\partial \alpha_1}{\partial x}$$

$$p_1 = p_1(\rho_1, e_1)$$

$$\frac{\partial \alpha_1 \rho_1 E_1}{\partial t} + \frac{\partial \alpha_1 u_1 (\rho_1 E_1 + p_1)}{\partial x} = p_I u_I \frac{\partial \alpha_1}{\partial x}$$

$$\frac{\partial \alpha_2 \rho_2}{\partial t} + \frac{\partial \alpha_2 \rho_2 u_2}{\partial x} = 0$$

$$\frac{\partial \alpha_2 \rho_2 u_2}{\partial t} + \frac{\partial \alpha_2 (\rho_2 u_2^2 + p_2)}{\partial x} = p_I \frac{\partial \alpha_2}{\partial x}$$

$$p_2 = p_2(\rho_2, e_2)$$

$$\frac{\partial \alpha_2 \rho_2 E_2}{\partial t} + \frac{\partial \alpha_2 u_2 (\rho_2 E_2 + p_2)}{\partial x} = p_I u_I \frac{\partial \alpha_2}{\partial x}$$

Deux choses manquent:
- p_I et u_I
- Les effets dissipatifs.

Ajout des effets dissipatifs

$$\frac{\partial \alpha_1}{\partial t} + u_I \frac{\partial \alpha_1}{\partial x} = \mu(p_1 - p_2)$$

Relaxation des pressions

$$\frac{\partial \alpha_1 \rho_1}{\partial t} + \frac{\partial \alpha_1 \rho_1 u_1}{\partial x} = 0$$

Relaxation des vitesses: force de traînée

$$\frac{\partial \alpha_1 \rho_1 u_1}{\partial t} + \frac{\partial \alpha_1 (\rho_1 u_1^2 + p_1)}{\partial x} = p_I \frac{\partial \alpha_1}{\partial x} + \lambda(u_2 - u_1)$$

Relaxation des températures = échange de chaleur

$$\frac{\partial \alpha_1 \rho_1 E_1}{\partial t} + \frac{\partial \alpha_1 u_1 (\rho_1 E_1 + p_1)}{\partial x} = p_I u_I \frac{\partial \alpha_1}{\partial x} + \lambda u_I (u_2 - u_1) - p_I \mu(p_1 - p_2) + H(T_2 - T_1)$$

Travail de la trainée

Travail des forces de pression interstitielles

$$\frac{\partial \alpha_2 \rho_2}{\partial t} + \frac{\partial \alpha_2 \rho_2 u_2}{\partial x} = 0$$

$$\frac{\partial \alpha_2 \rho_2 u_2}{\partial t} + \frac{\partial \alpha_2 (\rho_2 u_2^2 + p_2)}{\partial x} = p_I \frac{\partial \alpha_2}{\partial x} - \lambda(u_2 - u_1)$$

$$\frac{\partial \alpha_2 \rho_2 E_2}{\partial t} + \frac{\partial \alpha_2 u_2 (\rho_2 E_2 + p_2)}{\partial x} = p_I u_I \frac{\partial \alpha_2}{\partial x} - \lambda u_I (u_2 - u_1) + p_I \mu(p_1 - p_2) - H(T_2 - T_1)$$

Correlations

Les paramètres de relaxation sont donnés par des formules et corrélations:

1 = particules; 2 = gaz

$$A_I / V = \frac{3\alpha_1}{R_1}$$

$$\mu = \frac{A_I / V}{\rho_1 c_1 + \rho_2 c_2}$$

$$C_d = \frac{24}{R_e} + \frac{2.6\frac{R_e}{5}}{1 + \left(\frac{R_e}{5}\right)^{1.52}} + ...$$

$$\lambda = \frac{1}{2}\rho_2 \left|u_1 - u_2\right| C_d \frac{A_I}{V}$$

$$H = h_{conv} \frac{A_I}{V}$$

$$N_u = \frac{h_{conv} 2R_1}{\lambda_2}$$

$$N_u = 2 + 1.8 R_e{}^{1/2} P_{randlt}{}^{1/3}$$

$$R_e = \frac{\rho_2 \left|u_1 - u_2\right| 2R_1}{\mu_{2(gaz)}}$$

Exemple: Relaxation des vitesses (seule) en un point donné de l’espace

$$\frac{\partial \alpha_1}{\partial t} = 0$$

$$\frac{\partial \alpha_1 \rho_1}{\partial t} = 0 \qquad \frac{\partial \alpha_2 \rho_2}{\partial t} = 0$$

$$\frac{\partial \alpha_1 \rho_1 u_1}{\partial t} = \lambda(u_2 - u_1) \qquad \frac{\partial \alpha_2 \rho_2 u_2}{\partial t} = \lambda(u_1 - u_2)$$

$$\frac{\partial \alpha_1 \rho_1 E_1}{\partial t} = u_I \lambda(u_2 - u_1) \qquad \frac{\partial \alpha_2 \rho_2 E_2}{\partial t} = u_I \lambda(u_1 - u_2)$$

Relaxation des vitesses (seules)

$$\begin{tikzcd}
\frac{\partial \alpha_1}{\partial t}=0 \arrow[r] & \alpha_1=cst & \\
\frac{\partial \alpha_1\rho_1}{\partial t}=0 \arrow[r] & \rho_1=cst & \\
\frac{\partial \alpha_1\rho_1 u_1}{\partial t}=\lambda(u_2-u_1) \arrow[r] & \frac{\partial u_1}{\partial t}=\frac{\lambda(u_2-u_1)}{\alpha_1\rho_1} \arrow[dr] & \\
\frac{\partial \alpha_1\rho_1 E_1}{\partial t}=u_I\lambda(u_2-u_1) & & \frac{\partial (u_1-u_2)}{\partial t}=\left(\frac{1}{\alpha_1\rho_1}+\frac{1}{\alpha_2\rho_2}\right)\lambda(u_2-u_1) \\
\frac{\partial \alpha_2\rho_2}{\partial t}=0 \arrow[r] & \rho_2=cst & \\
\frac{\partial \alpha_2\rho_2 u_2}{\partial t}=\lambda(u_1-u_2) \arrow[r] & \frac{\partial u_2}{\partial t}=-\frac{\lambda(u_2-u_1)}{\alpha_2\rho_2} \arrow[uur] & \\
\frac{\partial \alpha_2\rho_2 E_2}{\partial t}=u_I\lambda(u_1-u_2) & &
\end{tikzcd}$$

$$\frac{\partial(u_1-u_2)}{\partial t}=\left(\frac{1}{\alpha_1\rho_1}+\frac{1}{\alpha_2\rho_2}\right)\lambda(u_2-u_1)$$

$$\frac{d(u_1-u_2)}{(u_1-u_2)}=-\lambda\left(\frac{1}{\alpha_1\rho_1}+\frac{1}{\alpha_2\rho_2}\right)dt$$

$$(u_1-u_2)=(u_1^{\,0}-u_2^{\,0})\exp\left(-\lambda\left(\frac{1}{\alpha_1\rho_1}+\frac{1}{\alpha_2\rho_2}\right)(t-t_0)\right)$$

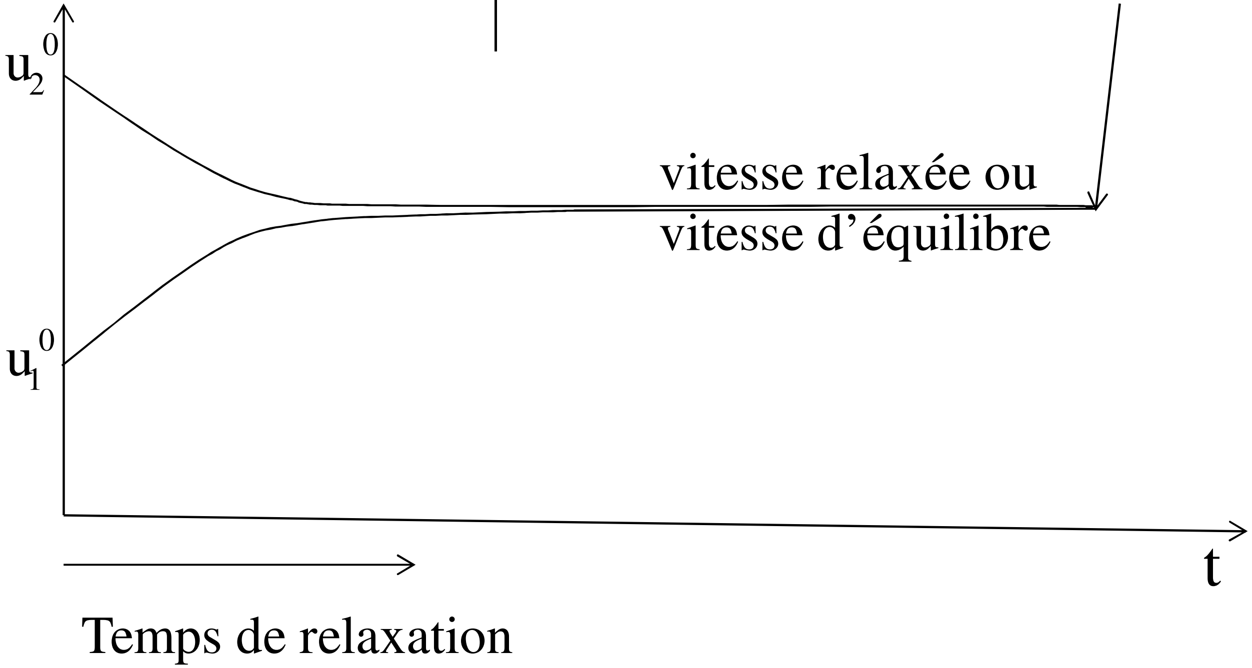
$$\begin{tikzcd}
\frac{\partial\alpha_1\rho_1u_1}{\partial t}=\lambda(u_2-u_1) \arrow[dr] & \\
& \frac{\partial\alpha_1\rho_1u_1+\alpha_2\rho_2u_2}{\partial t}=0 \\
\frac{\partial\alpha_2\rho_2u_2}{\partial t}=\lambda(u_1-u_2) \arrow[ur] &
\end{tikzcd}$$

$$\left(\alpha_1\rho_1u_1+\alpha_2\rho_2u_2\right)^0=\left(\alpha_1\rho_1u_1+\alpha_2\rho_2u_2\right)^*$$

$$u_1^*=u_2^*=u^*$$

$$u^*=\frac{\left(\alpha_1\rho_1u_1+\alpha_2\rho_2u_2\right)^0}{\left(\alpha_1\rho_1+\alpha_2\rho_2\right)^*}=\frac{\left(\alpha_1\rho_1u_1+\alpha_2\rho_2u_2\right)^0}{\left(\alpha_1\rho_1+\alpha_2\rho_2\right)^0}$$

Le même processus se produit pour les pressions et températures.



Il reste à déterminer p_I et u_I

Nous allons chercher des estimations ‘admissibles’ par rapport au respect de la seconde loi de la thermodynamique.

$$dS_{\text{système}} \geq 0$$

Comment utiliser ceci?

Il nous faut déterminer les équations d’évolution des entropies des phases puis les sommer pour étudier les termes de production d’entropie du système isolé.

Méthode

1) Ecrire toutes les équations en variables ‘physiques’:

   $\alpha_k ;\ u_k ;\ \rho_k ;\ e_k$

2) En déduire les équations d’évolution sur s_k

3) Sommer les équations d’entropie.

Ecriture des équations en variables 'physiques'

$$\frac{\partial \alpha_1}{\partial t} + u_I \frac{\partial \alpha_1}{\partial x} = \mu(p_1 - p_2)$$ OK

$$\frac{\partial \alpha_1 \rho_1}{\partial t} + \frac{\partial \alpha_1 \rho_1 u_1}{\partial x} = 0$$

$$\alpha_1 \frac{\partial \rho_1}{\partial t} + \rho_1 \frac{\partial \alpha_1}{\partial t} + \alpha_1 \rho_1 \frac{\partial u_1}{\partial x} + u_1 \frac{\partial \alpha_1 \rho_1}{\partial x} = 0$$

$$\alpha_1 \frac{\partial \rho_1}{\partial t} + \rho_1 \left(-u_I \frac{\partial \alpha_1}{\partial x} + \mu(p_1 - p_2) \right) + \alpha_1 \rho_1 \frac{\partial u_1}{\partial x} + \alpha_1 u_1 \frac{\partial \rho_1}{\partial x} + \rho_1 u_1 \frac{\partial \alpha_1}{\partial x} = 0$$

$$\frac{\partial \rho_1}{\partial t} + u_1 \frac{\partial \rho_1}{\partial x} + \frac{\rho_1 (u_1 - u_I)}{\alpha_1} \frac{\partial \alpha_1}{\partial x} + \rho_1 \frac{\partial u_1}{\partial x} = -\frac{\rho_1}{\alpha_1} \mu(p_1 - p_2)$$ OK

$$\frac{\partial \alpha_1 \rho_1 u_1}{\partial t} + \frac{\partial \alpha_1 (\rho_1 u_1^2 + p_1)}{\partial x} = p_I \frac{\partial \alpha_1}{\partial x} + \lambda(u_2 - u_1)$$

$$\alpha_1 \rho_1 \frac{\partial u_1}{\partial t} + u_1 \frac{\partial \alpha_1 \rho_1}{\partial t} + \alpha_1 \rho_1 u_1 \frac{\partial u_1}{\partial x} + u_1 \frac{\partial \alpha_1 \rho_1 u_1}{\partial x} + \alpha_1 \frac{\partial p_1}{\partial x} + p_1 \frac{\partial \alpha_1}{\partial x} = p_I \frac{\partial \alpha_1}{\partial x} + \lambda(u_2 - u_1)$$

$$\alpha_1 \rho_1 \left(\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} \right) + \alpha_1 \frac{\partial p_1}{\partial x} + (p_1 - p_I) \frac{\partial \alpha_1}{\partial x} = \lambda(u_2 - u_1)$$

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + \frac{1}{\rho_1} \frac{\partial p_1}{\partial x} + \frac{(p_1 - p_I)}{\alpha_1 \rho_1} \frac{\partial \alpha_1}{\partial x} = \frac{\lambda(u_2 - u_1)}{\alpha_1 \rho_1}$$

OK

$$\frac{\partial\alpha_1\rho_1E_1}{\partial t}+\frac{\partial\alpha_1u_1(\rho_1E_1+p_1)}{\partial x}=p_Iu_I\frac{\partial\alpha_1}{\partial x}+\lambda u_I(u_2-u_1)-p_I\mu(p_1-p_2)+H(T_2-T_1)$$

$$\alpha_1\rho_1\frac{\partial E_1}{\partial t}+E_1\frac{\partial\alpha_1\rho_1}{\partial t}+\alpha_1\rho_1u_1\frac{\partial E_1}{\partial x}+E_1\frac{\partial\alpha_1\rho_1u_1}{\partial x}+\alpha_1u_1\frac{\partial p_1}{\partial x}+p_1\frac{\partial\alpha_1u_1}{\partial x}=$$
$$p_Iu_I\frac{\partial\alpha_1}{\partial x}+\lambda u_I(u_2-u_1)-p_I\mu(p_1-p_2)+H(T_2-T_1)$$

$$\alpha_1\rho_1\left(\frac{\partial E_1}{\partial t}+u_1\frac{\partial E_1}{\partial x}\right)+\alpha_1u_1\frac{\partial p_1}{\partial x}+(p_1u_1-p_Iu_I)\frac{\partial\alpha_1}{\partial x}+\alpha_1p_1\frac{\partial u_1}{\partial x}=$$
$$\lambda u_I(u_2-u_1)-p_I\mu(p_1-p_2)+H(T_2-T_1)$$

$$\alpha_1\rho_1\left(\frac{\partial e_1}{\partial t}+u_1\frac{\partial e_1}{\partial x}\right)+\alpha_1\rho_1u_1\left(\frac{\partial u_1}{\partial t}+u_1\frac{\partial u_1}{\partial x}\right)+\alpha_1u_1\frac{\partial p_1}{\partial x}+(p_1u_1-p_Iu_I)\frac{\partial\alpha_1}{\partial x}+\alpha_1p_1\frac{\partial u_1}{\partial x}=$$
$$\lambda u_I(u_2-u_1)-p_I\mu(p_1-p_2)+H(T_2-T_1)$$

$$\alpha_1\rho_1\left(\frac{\partial e_1}{\partial t}+u_1\frac{\partial e_1}{\partial x}\right)+\alpha_1\rho_1 u_1\left(\frac{\partial u_1}{\partial t}+u_1\frac{\partial u_1}{\partial x}\right)+\alpha_1 u_1\frac{\partial p_1}{\partial x}+(p_1u_1-p_I u_I)\frac{\partial \alpha_1}{\partial x}+\alpha_1 p_1\frac{\partial u_1}{\partial x}=$$
$$\lambda u_I(u_2-u_1)-p_I\mu(p_1-p_2)+H(T_2-T_1)$$

Or, qdm $$\frac{\partial u_1}{\partial t}+u_1\frac{\partial u_1}{\partial x}=-\frac{1}{\rho_1}\frac{\partial p_1}{\partial x}-\frac{(p_1-p_I)}{\alpha_1\rho_1}\frac{\partial \alpha_1}{\partial x}+\frac{\lambda(u_2-u_1)}{\alpha_1\rho_1}$$

$$\alpha_1\rho_1\left(\frac{\partial e_1}{\partial t}+u_1\frac{\partial e_1}{\partial x}\right)+p_I\left(u_1-u_I\right)\frac{\partial \alpha_1}{\partial x}+\alpha_1 p_1\frac{\partial u_1}{\partial x}$$
$$=\lambda(u_I-u_1)(u_2-u_1)-p_I\mu(p_1-p_2)+H(T_2-T_1)$$

Or, masse
$$\frac{\partial \rho_1}{\partial t}+u_1\frac{\partial \rho_1}{\partial x}+\frac{\rho_1\left(u_1-u_I\right)}{\alpha_1}\frac{\partial \alpha_1}{\partial x}+\rho_1\frac{\partial u_1}{\partial x}=-\frac{\rho_1}{\alpha_1}\mu(p_1-p_2)$$
$$\frac{\partial u_1}{\partial x}=-\frac{1}{\rho_1}\left(\frac{\partial \rho_1}{\partial t}+u_1\frac{\partial \rho_1}{\partial x}\right)-\frac{\left(u_1-u_I\right)}{\alpha_1}\frac{\partial \alpha_1}{\partial x}-\frac{1}{\alpha_1}\mu(p_1-p_2)$$

$$\alpha_1\rho_1\left[\left(\frac{\partial e_1}{\partial t}+u_1\frac{\partial e_1}{\partial x}\right)-\frac{p_1}{\rho_1^2}\left(\frac{\partial \rho_1}{\partial t}+u_1\frac{\partial \rho_1}{\partial x}\right)\right]+(p_I-p_1)\left(u_1-u_I\right)\frac{\partial \alpha_1}{\partial x}$$
$$=\lambda(u_I-u_1)(u_2-u_1)+\mu(p_1-p_2)(p_1-p_I)+H(T_2-T_1)$$

$$\alpha_1\rho_1\left[\left(\frac{\partial e_1}{\partial t}-\frac{p_1}{\rho_1^2}\frac{\partial \rho_1}{\partial t}\right)+u_1\left(\frac{\partial e_1}{\partial x}-\frac{p_1}{\rho_1^2}\frac{\partial \rho_1}{\partial x}\right)\right]$$
$$=(p_I-p_1)\left(u_I-u_1\right)\frac{\partial \alpha_1}{\partial x}+\lambda(u_I-u_1)(u_2-u_1)+\mu(p_1-p_2)(p_1-p_I)+H(T_2-T_1)$$

Or, $v_1=\frac{1}{\rho_1}$ et $\frac{\partial v_1}{\partial t}=-\frac{1}{\rho_1^2}\frac{\partial \rho_1}{\partial t}$

$$\alpha_1\rho_1\left[\left(\frac{\partial e_1}{\partial t}+p_1\frac{\partial v_1}{\partial t}\right)+u_1\left(\frac{\partial e_1}{\partial x}+p_1\frac{\partial v_1}{\partial x}\right)\right]$$
$$=(p_I-p_1)\left(u_I-u_1\right)\frac{\partial \alpha_1}{\partial x}+\lambda(u_I-u_1)(u_2-u_1)+\mu(p_1-p_2)(p_1-p_I)+H(T_2-T_1)$$

$$\alpha_1\rho_1\left[\left(\frac{\partial e_1}{\partial t}+p_1\frac{\partial v_1}{\partial t}\right)+u_1\left(\frac{\partial e_1}{\partial x}+p_1\frac{\partial v_1}{\partial x}\right)\right]$$

$$=(p_I-p_1)\left(u_I-u_1\right)\frac{\partial\alpha_1}{\partial x}+\lambda(u_I-u_1)(u_2-u_1)+\mu(p_1-p_2)(p_1-p_I)+H(T_2-T_1)$$

Or, la première loi de la thermodynamique, ou identité de Gibbs nous informe que: $de=Tds-pdv$

$$Tds=de+pdv$$

Ainsi,

$$\alpha_1\rho_1 T_1\left[\frac{\partial s_1}{\partial t}+u_1\frac{\partial s_1}{\partial x}\right]$$

$$=(p_I-p_1)\left(u_I-u_1\right)\frac{\partial\alpha_1}{\partial x}+\lambda(u_I-u_1)(u_2-u_1)+\mu(p_1-p_2)(p_1-p_I)+H(T_2-T_1)$$

OK

$$\alpha_1\rho_1 T_1\left[\frac{\partial s_1}{\partial t}+u_1\frac{\partial s_1}{\partial x}\right]$$

$$=(p_I-p_1)\left(u_I-u_1\right)\frac{\partial\alpha_1}{\partial x}+\lambda(u_I-u_1)(u_2-u_1)+\mu(p_1-p_2)(p_1-p_I)+H(T_2-T_1)$$

$$\alpha_2\rho_2 T_2\left[\frac{\partial s_2}{\partial t}+u_2\frac{\partial s_2}{\partial x}\right]$$

$$=(p_I-p_2)\left(u_I-u_2\right)\frac{\partial\alpha_2}{\partial x}+\lambda(u_I-u_2)(u_1-u_2)+\mu(p_2-p_1)(p_2-p_I)+H(T_1-T_2)$$

Somme,

$$\alpha_1\rho_1\left[\frac{\partial s_1}{\partial t}+u_1\frac{\partial s_1}{\partial x}\right]+\alpha_2\rho_2\left[\frac{\partial s_2}{\partial t}+u_2\frac{\partial s_2}{\partial x}\right]$$

$$=\frac{(p_I-p_1)\left(u_I-u_1\right)}{T_1}\frac{\partial\alpha_1}{\partial x}+\frac{\lambda(u_I-u_1)(u_2-u_1)}{T_1}+\frac{\mu(p_1-p_2)(p_1-p_I)}{T_1}+\frac{H(T_2-T_1)}{T_1}$$

$$-\frac{(p_I-p_2)\left(u_I-u_2\right)}{T_2}\frac{\partial\alpha_1}{\partial x}+\frac{\lambda(u_I-u_2)(u_1-u_2)}{T_2}+\frac{\mu(p_2-p_1)(p_2-p_I)}{T_2}+\frac{H(T_1-T_2)}{T_2}$$

Entropie du système isolé

$$\frac{\partial\left(\alpha_1\rho_1 s_1+\alpha_2\rho_2 s_2\right)}{\partial t}+\frac{\partial\left(\alpha_1\rho_1 u_1 s_1+\alpha_2\rho_2 u_2 s_2\right)}{\partial x}$$

$$=\left[\frac{(p_I-p_1)\left(u_I-u_1\right)}{T_1}-\frac{(p_I-p_2)\left(u_I-u_2\right)}{T_2}\right]\frac{\partial\alpha_1}{\partial x}$$

$$+\lambda(u_1-u_2)\left[\frac{(u_I-u_2)}{T_2}-\frac{(u_I-u_1)}{T_1}\right]+\mu(p_1-p_2)\left[\frac{(p_1-p_I)}{T_1}-\frac{(p_2-p_1)}{T_2}\right]$$

$$+H(T_1-T_2)\left[\frac{1}{T_2}-\frac{1}{T_1}\right]\geq 0$$

Seconde loi de la thermo, vérifiée ou pas ?

Etude de la production d’entropie

$$\frac{\partial\left(\alpha_1\rho_1 s_1+\alpha_2\rho_2 s_2\right)}{\partial t}+\frac{\partial\left(\alpha_1\rho_1 u_1 s_1+\alpha_2\rho_2 u_2 s_2\right)}{\partial x}$$

$$=\left[\frac{(p_I-p_1)\left(u_I-u_1\right)}{T_1}-\frac{(p_I-p_2)\left(u_I-u_2\right)}{T_2}\right]\frac{\partial\alpha_1}{\partial x}$$

$$+\lambda(u_1-u_2)\left[\frac{(u_I-u_2)}{T_2}-\frac{(u_I-u_1)}{T_1}\right]+\mu(p_1-p_2)\left[\frac{(p_1-p_I)}{T_1}-\frac{(p_2-p_1)}{T_2}\right]$$

$$+H(T_1-T_2)\left[\frac{T_1-T_2}{T_1T_2}\right]\geq 0$$

Option 1 : $p_I=p_1;\quad u_I=u_2$

$$\frac{\partial\left(\alpha_1\rho_1 s_1+\alpha_2\rho_2 s_2\right)}{\partial t}+\frac{\partial\left(\alpha_1\rho_1 u_1 s_1+\alpha_2\rho_2 u_2 s_2\right)}{\partial x}=\lambda\frac{(u_1-u_2)^2}{T_1}+\mu\frac{(p_1-p_2)^2}{T_2}+H\frac{(T_1-T_2)^2}{T_1T_2}\geq 0$$

Option 2 : $p_I=p_2;\quad u_I=u_1$

$$\frac{\partial\left(\alpha_1\rho_1 s_1+\alpha_2\rho_2 s_2\right)}{\partial t}+\frac{\partial\left(\alpha_1\rho_1 u_1 s_1+\alpha_2\rho_2 u_2 s_2\right)}{\partial x}=\lambda\frac{(u_1-u_2)^2}{T_2}+\mu(p_1-p_2)^2\left[\frac{1}{T_1}+\frac{1}{T_2}\right]+H\frac{(T_1-T_2)^2}{T_1T_2}\geq 0$$

Modèle d'écoulement diphasique admissible

$$\frac{\partial \alpha_1}{\partial t} + u_2 \frac{\partial \alpha_1}{\partial x} = \mu(p_1 - p_2)$$

$$\mu = \frac{A_I / V}{\rho_1 c_1 + \rho_2 c_2}$$

$$A_I / V = \frac{3\alpha_1}{R_1}$$

$$\frac{\partial \alpha_1 \rho_1}{\partial t} + \frac{\partial \alpha_1 \rho_1 u_1}{\partial x} = 0$$

$$\lambda = \frac{1}{2} \rho_2 \left| u_1 - u_2 \right| C_d \frac{A_I}{V}$$

$$\frac{\partial \alpha_1 \rho_1 u_1}{\partial t} + \frac{\partial \alpha_1 (\rho_1 u_1^2 + p_1)}{\partial x} = p_1 \frac{\partial \alpha_1}{\partial x} + \lambda(u_2 - u_1)$$

$$H = h_{conv} \frac{A_I}{V}$$

$$\frac{\partial \alpha_1 \rho_1 E_1}{\partial t} + \frac{\partial \alpha_1 u_1 (\rho_1 E_1 + p_1)}{\partial x} = p_1 u_2 \frac{\partial \alpha_1}{\partial x} + \lambda u_I (u_2 - u_1) - p_1 \mu (p_1 - p_2) + H(T_2 - T_1)$$

$$\frac{\partial \alpha_2 \rho_2}{\partial t} + \frac{\partial \alpha_2 \rho_2 u_2}{\partial x} = 0$$

$$\frac{\partial \alpha_2 \rho_2 u_2}{\partial t} + \frac{\partial \alpha_2 (\rho_2 u_2^2 + p_2)}{\partial x} = p_1 \frac{\partial \alpha_2}{\partial x} - \lambda(u_2 - u_1)$$

$$\frac{\partial \alpha_2 \rho_2 E_2}{\partial t} + \frac{\partial \alpha_2 u_2 (\rho_2 E_2 + p_2)}{\partial x} = p_1 u_2 \frac{\partial \alpha_2}{\partial x} - \lambda u_2 (u_2 - u_1) + p_1 \mu (p_1 - p_2) - H(T_2 - T_1)$$

Hyperbolicité ?

Lorsque des phénomènes de propagation sont présents dans la physique du problème étudié (propagation du son par exemple), les équations doivent être hyperboliques (les termes sources sont supprimés dans cette analyse).

$$\frac{\partial W}{\partial t} + A(W)\frac{\partial W}{\partial x} = 0$$

Les valeurs propres de la matrice A(W) doivent être réelles.

Sinon, modèle mal posé.

Le choix des variables dans W importe peu.

Mais certains choix permettent des calculs plus rapides.

$$W = \begin{pmatrix} s_1 \\ s_2 \\ \alpha_1 \\ \rho_1 \\ \rho_2 \\ u_1 \\ u_2 \end{pmatrix}$$

$$\frac{\partial s_1}{\partial t} + u_1 \frac{\partial s_1}{\partial x} = 0$$

$$\frac{\partial s_2}{\partial t} + u_2 \frac{\partial s_2}{\partial x} = 0$$

$$\frac{\partial \alpha_1}{\partial t} + u_2 \frac{\partial \alpha_1}{\partial x} = 0$$

$$\frac{\partial \rho_1}{\partial t} + u_1 \frac{\partial \rho_1}{\partial x} + \frac{\rho_1 \left(u_1 - u_2\right)}{\alpha_1} \frac{\partial \alpha_1}{\partial x} + \rho_1 \frac{\partial u_1}{\partial x} = 0$$

$$\frac{\partial \rho_2}{\partial t} + u_2 \frac{\partial \rho_2}{\partial x} - \frac{\rho_2 \left(u_2 - u_1\right)}{\alpha_2} \frac{\partial \alpha_1}{\partial x} + \rho_2 \frac{\partial u_2}{\partial x} = 0$$

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + \frac{1}{\rho_1} \frac{\partial p_1}{\partial x} = 0$$

$$\frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial x} + \frac{1}{\rho_2} \frac{\partial p_2}{\partial x} - \frac{(p_2 - p_I)}{\alpha_2 \rho_2} \frac{\partial \alpha_1}{\partial x} = 0$$

$$W = \begin{pmatrix} s_1 \\ s_2 \\ \alpha_1 \\ \rho_1 \\ \rho_2 \\ u_1 \\ u_2 \end{pmatrix}$$

Problème ici.
p_1 et p_2 ne font pas
partie du vecteur W.

$$\frac{\partial s_1}{\partial t} + u_1 \frac{\partial s_1}{\partial x} = 0$$

$$\frac{\partial s_2}{\partial t} + u_2 \frac{\partial s_2}{\partial x} = 0$$

$$\frac{\partial \alpha_1}{\partial t} + u_2 \frac{\partial \alpha_1}{\partial x} = 0$$

$$\frac{\partial \rho_1}{\partial t} + u_1 \frac{\partial \rho_1}{\partial x} + \frac{\rho_1 \left(u_1 - u_2\right)}{\alpha_1} \frac{\partial \alpha_1}{\partial x} + \rho_1 \frac{\partial u_1}{\partial x} = 0$$

$$\frac{\partial \rho_2}{\partial t} + u_2 \frac{\partial \rho_2}{\partial x} - \frac{\rho_2 \left(u_2 - u_1\right)}{\alpha_2} \frac{\partial \alpha_1}{\partial x} + \rho_2 \frac{\partial u_2}{\partial x} = 0$$

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + \frac{1}{\rho_1}\left(\left(\frac{\partial p_1}{\partial \rho_1}\right)_{s_1} \frac{\partial \rho_1}{\partial x} + \left(\frac{\partial p_1}{\partial s_1}\right)_{\rho_1} \frac{\partial s_1}{\partial x} \right) = 0$$

$$\frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial x} + \frac{1}{\rho_2}\left(\left(\frac{\partial p_2}{\partial \rho_2}\right)_{s_2} \frac{\partial \rho_2}{\partial x} + \left(\frac{\partial p_2}{\partial s_2}\right)_{\rho_2} \frac{\partial s_2}{\partial x} \right) - \frac{(p_2 - p_I)}{\alpha_2 \rho_2} \frac{\partial \alpha_1}{\partial x} = 0$$

$$W = \begin{pmatrix} s_1 \\ s_2 \\ \alpha_1 \\ \rho_1 \\ \rho_2 \\ u_1 \\ u_2 \end{pmatrix}$$

$$\frac{\partial s_1}{\partial t} + u_1 \frac{\partial s_1}{\partial x} = 0$$

$$\frac{\partial s_2}{\partial t} + u_2 \frac{\partial s_2}{\partial x} = 0$$

$$\frac{\partial \alpha_1}{\partial t} + u_2 \frac{\partial \alpha_1}{\partial x} = 0$$

$$\frac{\partial \rho_1}{\partial t} + u_1 \frac{\partial \rho_1}{\partial x} + \frac{\rho_1 \left(u_1 - u_2\right)}{\alpha_1} \frac{\partial \alpha_1}{\partial x} + \rho_1 \frac{\partial u_1}{\partial x} = 0$$

$$\frac{\partial \rho_2}{\partial t} + u_2 \frac{\partial \rho_2}{\partial x} - \frac{\rho_2 \left(u_2 - u_1\right)}{\alpha_2} \frac{\partial \alpha_1}{\partial x} + \rho_2 \frac{\partial u_2}{\partial x} = 0$$

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + \frac{c_1^2}{\rho_1} \frac{\partial \rho_1}{\partial x} + \left. \frac{1}{\rho_1} \frac{\partial p_1}{\partial s_1} \right)_{\rho_1} \frac{\partial s_1}{\partial x} = 0$$

$$\frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial x} + \left(\frac{c_2^2}{\rho_2} \frac{\partial \rho_2}{\partial x} + \left. \frac{1}{\rho_2} \frac{\partial p_2}{\partial s_2} \right)_{\rho_2} \frac{\partial s_2}{\partial x} \right) - \frac{(p_2 - p_I)}{\alpha_2 \rho_2} \frac{\partial \alpha_1}{\partial x} = 0$$

$$W = \begin{pmatrix} s_1 \\ s_2 \\ \alpha_1 \\ \rho_1 \\ \rho_2 \\ u_1 \\ u_2 \end{pmatrix}$$

$$A(W) = \begin{pmatrix}
u_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & u_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & u_2 & 0 & 0 & 0 & 0 \\
0 & 0 & \dfrac{\rho_1 \left(u_1 - u_2\right)}{\alpha_1} & u_1 & 0 & \rho_1 & 0 \\
0 & 0 & -\dfrac{\rho_2 \left(u_2 - u_1\right)}{\alpha_2} & 0 & u_2 & 0 & \rho_2 \\
\left. \dfrac{1}{\rho_1} \dfrac{\partial p_1}{\partial s_1} \right)_{\rho_1} & 0 & 0 & \dfrac{c_1^2}{\rho_1} & 0 & u_1 & 0 \\
0 & \left. \dfrac{1}{\rho_2} \dfrac{\partial p_2}{\partial s_2} \right)_{\rho_2} & -\dfrac{(p_2 - p_I)}{\alpha_2 \rho_2} & 0 & \dfrac{c_2^2}{\rho_2} & 0 & u_2
\end{pmatrix}$$

$$\det(A-\lambda I)=\begin{pmatrix} u_1-\lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & u_2-\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & u_2-\lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{\rho_1(u_1-u_2)}{\alpha_1} & u_1-\lambda & 0 & \rho_1 & 0 \\ 0 & 0 & -\dfrac{\rho_2(u_2-u_1)}{\alpha_2} & 0 & u_2-\lambda & 0 & \rho_2 \\ \left.\dfrac{1}{\rho_1}\dfrac{\partial p_1}{\partial s_1}\right)_{\rho_1} & 0 & 0 & \dfrac{c_1^2}{\rho_1} & 0 & u_1-\lambda & 0 \\ 0 & \left.\dfrac{1}{\rho_2}\dfrac{\partial p_2}{\partial s_2}\right)_{\rho_2} & -\dfrac{(p_2-p_I)}{\alpha_2\rho_2} & 0 & \dfrac{c_2^2}{\rho_2} & 0 & u_2-\lambda \end{pmatrix}=0$$

$\lambda_1=u_1;\lambda_2=u_2;\lambda_3=u_2;$ puis

$$\begin{vmatrix} u_1-\lambda & 0 & \rho_1 & 0 \\ 0 & u_2-\lambda & 0 & \rho_2 \\ \dfrac{c_1^2}{\rho_1} & 0 & u_1-\lambda & 0 \\ 0 & \dfrac{c_2^2}{\rho_2} & 0 & u_2-\lambda \end{vmatrix}=0$$

$$\begin{vmatrix} u_1-\lambda & 0 & \rho_1 & 0 \\ 0 & u_2-\lambda & 0 & \rho_2 \\ \dfrac{c_1^2}{\rho_1} & 0 & u_1-\lambda & 0 \\ 0 & \dfrac{c_2^2}{\rho_2} & 0 & u_2-\lambda \end{vmatrix} = 0 \quad \Leftrightarrow (u_1-\lambda) \begin{vmatrix} u_2-\lambda & 0 & \rho_2 \\ 0 & u_1-\lambda & 0 \\ \dfrac{c_2^2}{\rho_2} & 0 & u_2-\lambda \end{vmatrix} + \rho_1 \begin{vmatrix} 0 & u_2-\lambda & \rho_2 \\ \dfrac{c_1^2}{\rho_1} & 0 & 0 \\ 0 & \dfrac{c_2^2}{\rho_2} & u_2-\lambda \end{vmatrix} = 0$$

$$(u_1-\lambda)\left[(u_2-\lambda)^2(u_1-\lambda)-(u_1-\lambda)c_2^2\right]+\rho_1\left[\frac{c_1^2}{\rho_1}\frac{c_2^2}{\rho_2}\rho_2-(u_2-\lambda)^2\frac{c_1^2}{\rho_1}\right]=0$$

$$(u_1-\lambda)^2\left[(u_2-\lambda)^2-c_2^2\right]+c_1^2\left[c_2^2-(u_2-\lambda)^2\right]=0$$

$$\left[(u_2-\lambda)^2-c_2^2\right]\left[(u_1-\lambda)^2-c_1^2\right]=0$$

$$\lambda_4=u_1+c_1;\lambda_5=u_1-c_1;\lambda_6=u_2-c_2;\lambda_7=u_2+c_2$$

Le système est donc hyperbolique et admet pour vitesses d’ondes:

$$\lambda_1=u_1;\lambda_2=u_2;\lambda_3=u_2;\ \lambda_4=u_1+c_1;\lambda_5=u_1-c_1;\lambda_6=u_2-c_2;\lambda_7=u_2+c_2$$

Le modèle de la page 21 s'impose peu à peu dans tous les outils de simulation d'écoulements diphasiques:

- du domaine pétrolier,
- thermohydraulique des centrales nucléaires,
- explosions, chocs et détonations dans les fluides hétérogènes.

Chaque fois qu'un mélange diphasique à deux vitesses et deux températures est présent.